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# Application of a generalized bottleneck routing problem to the task of adhering to acceptable doses of radiation during the dismantling of radiation hazardous objects ⋆

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**Abstract:** The paper considers the bottleneck routing problem, the prototype of which is the task of minimizing the radiation dose of workers during the performance of work related to disassembly of radiation-hazardous equipment. The authors proposed an optimal algorithm for solving the problem, which is based on the dynamic programming method. Its effectiveness is illustrated by example.



*Keywords:* the task of minimizing radiation doses, routing problem, dynamic programming.

## 1. INTRODUCTION

The studies carried out in the article are motivated by the following substantive statement of the problem. In a certain area there are N point radiation hazardous objects that must be dismantled ("turned off") and decontaminated contaminated territory. To perform this work is determined by a certain team of workers. On the first day they dismantle the first object. At the same time they receive a dose of radiation from all existing facilities. After completing the work team leaves to rest. The next day, this team goes to the dismantling of the next object. Every day, work begins at the place of completion of work on the previous day (shift), that is, the dose costs for moving to this point are not taken into account (it can be assumed that workers arrive at this point in a car protected from radiation). At the same time, workers receive exposure from all remaining objects. After performing the work, the object being dismounted ("switched off") ceases to be a source of radiation. On the $N$-th day, the brigade dismantles the last object. When determining the doses of radiation specialists during the work of the recommendations used IAEA (2000). When performing these works, each worker and the team as a whole receive a certain dose of radiation daily. According to regulatory documents, daily dose budgets per employee are established, which should not be exceeded. In accordance with the recommendations of the International Commission on Radiological Protection (ICRP) ICRP (2007) on the implementation of the principle of optimization of radiation protection, it is advisable to determine among all the options for ordering the dismantling of facilities to be optimal at which the maximum dose for each of the $N$ working days will be minimal.

Related issues were addressed in the monograph Korobkin (2012), but an additive quality criterion was considered there.

The problem considered in the work is a generalization of the well-known task of the traveling salesman problem (TSP); in this connection see Melamed (1989a,b,c); Gutin (2002); Cook (2012); Gimadi (2016). In TSP the branch-and-bound method is used very widely (see Little (1963)). Two variants of dynamic programming (DP) for TSP are reduced in Bellman (1958); Held, Karp (1962). We use the DP-procedure similar Chentsov (2016). The new appearance of our investigation is connected with optimization of the initial state. In addition we use the economical DP-procedure introduced in (Chentsov , 2008, §4.9). For this procedure, total number of values of Bellman function is not required. We use only layers of this function (in the case when precedence conditions exists). The singularity connected with optimization of the initial state is realized only on last step of the Bellman function construction.

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## 2. DEFINITIONS AND DESIGNATIONS

In our constructions, we use quantors and propositional connectives; $\emptyset$ is empty set, $:=$ is equality by definition. For any objects $p$ and $q$, by $\{p;q\}$ we denote the set containing $p$, $q$ and not containing none other objects. Then Kuratovskii (1970), for any objects $\alpha$ and $\beta$, in the form of $(\alpha,\beta) := \{\{\alpha\};\{\alpha;\beta\}\}$, we have the ordered pair (OP) with first element $\alpha$ and second element $\beta$. For every OP $z$, by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ we denote first and second elements of $z$ respectively. If $\alpha,\beta$, and $\gamma$ are objects, then $(\alpha,\beta,\gamma) := ((\alpha,\beta),\gamma)$. For any sets $A, B$ and $C$, as usual, we suppose (see Dieudonne (1960)) $A\times B\times C =: (A\times B)\times C$.

If $H$ is a set, then $\mathcal{P}(H)$ is the nonempty family of all subsets of $H$ and $\mathcal{P}'(H) := \mathcal{P}(H)\setminus\{\emptyset\}$; $\mathrm{Fin}(H)$ is the family of all finite sets of $\mathcal{P}'(H)$. So, $\mathrm{Fin}(H)$ is the family of all nonempty finite subsets of $H$. For every nonempty sets $A$ and $B$, by $B^A$ we denote the set of all mappings from $A$ into $B$; if $h\in B^A$ and $C\in\mathcal{P}(A)$, then $h^1(C) := \{h(x) : x\in C\}$ is image of $C$ under operation of $h$. If $A, B, C$ and $D$ are nonempty sets, $h\in D^{A\times B\times C}$, $\mu\in A\times B$, and $\nu\in C$, then $h(\mu,\nu)$ is the value of $h$ at the point $(\mu,\nu)\in A\times B\times C$; of course, $h(\mu,\nu) = h(\mathrm{pr}_1(\mu),\mathrm{pr}_2(\mu),\nu)$. As usual, permutation of a nonempty set $H$ is a bijection of $H$ onto itself; if $\alpha$ is a permutation of $H$, then the permutation $\alpha^{-1}$ (of $H$) inverse to $\alpha$ is defined by following conditions

$$\alpha(\alpha^{-1}(h)) := \alpha^{-1}(\alpha(h)) = h \quad \forall h\in H.$$

Let $\mathbf{R}_+ =: \{\xi\in\mathbf{R}|0\le\xi\}$, where $\mathbf{R}$ is real line. Moreover, we suppose that $\mathbf{N} =: \{1;2;...\}$ and $\mathbf{N}_0 := \{0;1;2;...\}$. For $p\in\mathbf{N}_0$ and $q\in\mathbf{N}_0$, we introduce interval in $\mathbf{N}_0$ :

$$\overline{p,q} := \{k\in\mathbf{N}_0 \mid (p\le k)\&(k\le q)\}.$$

If $K$ is a nonempty finite set, then by definition $|K|\in\mathbf{N}$ is cardinality of $K$ and $(\mathrm{bi})[K]$ is the nonempty set of all bijections from $\overline{1,|K|}$ onto $K$. As usual, $|\emptyset| := 0$. For every nonempty set $S$, by $\mathcal{R}_+[S]$ we denote the set of all functions from $S$ into $\mathbf{R}_+$ : $\mathcal{R}_+[S] := (\mathbf{R}_+)^S$. In this article all considered real-valued functions are non-negative, i.e. there functions are elements of sets similar $\mathcal{R}_+[S]$.

## 3. MATHEMATICAL STATEMENT OF PROBLEM

This section presents a rigorous formulation of non-additive routing problem. But first, we will provide some meaningful setting focused on applications related to the dismantling of radiation-hazardous equipment or the cleaning of a territory contaminated as a result of an accident from point-like radiation-hazardous objects. Namely, we will consider the issue of organizing work over a long period of time on the decontamination of an emergency zone contaminated with radioactive objects (for simplicity, we take radioactive objects in the form of point sources). It is required to determine the procedure for performing day work cycles in order to deactivate a number of objects in a radiation-dangerous area. In this case, the daily dose of exposure of each employee should not exceed a certain established limit. For this, we will use the bottleneck routing problem.

We fix a nonempty set $X$ and its subset $X^0\in\mathcal{P}'(X)$; so, $X^0\ne\emptyset$ and $X^0\subset X$. Elements of $X^0$ can be used as initial states of our problem. Moreover, we fix $N$ sets

$$M_1\in\mathrm{Fin}(X), ..., M_N\in\mathrm{Fin}(X)$$

with the following properties $(M_p\cap M_q = \emptyset\ \forall p\in\overline{1,N}\ \forall q\in\overline{1,N}\setminus\{p\})\&(X^0\cap M_j=\emptyset\ \forall j\in\overline{1,N})$.

Moreover, fix $N$ relations

$$\mathbf{M}_1\in\mathcal{P}'(M_1\times M_1), ..., \mathbf{M}_N\in\mathcal{P}'(M_N\times M_N).$$

So, $\mathbf{M}_j\ne\emptyset$ and $\mathbf{M}_j\subset M_j\times M_j$ under $j\in\overline{1,N}$,; in addition, by $z\in\mathbf{M}_j$ we have a variant of (interior) works under visiting of $M_j$. Let $\mathbf{P} := (\mathrm{bi})[\overline{1,N}]$. If $x\in X^0$, then we can consider processes of the form

$$z_0 \longrightarrow z_1\in\mathbf{M}_{\alpha(1)} \longrightarrow ... \longrightarrow z_N\in\mathbf{M}_{\alpha(N)}, \qquad (1)$$

where $z_0=(x,x)$ and $\alpha\in\mathbf{P}$. Of course, (1) is a consolidated process. Every visiting $z_j\in\mathbf{M}_{\alpha(j)}$ is representated as the movement from $\mathrm{pr}_1(z_j)$ into $\mathrm{pr}_2(z_j)$ with realization of interior works connected with $M_j$. In the following, elements of $\mathbf{P}$ are called routes. The choice of the concrete route must devote to constraints in the form precedence conditions. In this connection, we fix the set $\mathbf{K}\in\mathcal{P}(\overline{1,N}\times\overline{1,N})$. Elements of $\mathbf{K}$ are called address pairs of indexes (the case $\mathbf{K}=\emptyset$ is not excluded). We suppose that

$$\forall\mathbf{K}_0\in\mathcal{P}'(\mathbf{K})\,\exists z_0\in\mathbf{K}_0 : \mathrm{pr}_1(z_0)\ne\mathrm{pr}_2(z) \quad \forall z\in\mathbf{K}_0; \ (2)$$

in (Chentsov , 2008, part 2) concrete variants of setting with condition (2) are reduced. For every pair $(i,j)\in\mathbf{K}$, the megalopolis $M_i$ must be visit before $M_j$ (were introduced precedence conditions). Then

$$\mathbf{A} := \{\alpha\in\mathbf{P}\,|\,\forall z\in\mathbf{K}\forall t_1\in\overline{1,N}\forall t_2\in\overline{1,N}$$
$$((\alpha(t_1)=\mathrm{pr}_1(z))\&(\alpha(t_2)=\mathrm{pr}_2(z)))\Rightarrow(t_1<t_2)\}$$
$$= \{\alpha\in\mathbf{P}\,|\,\alpha^{-1}(\mathrm{pr}_1(z))<\alpha^{-1}(\mathrm{pr}_2(z)) \quad \forall z\in\mathbf{K}\}$$

is the set of all admissible routes. According to (2), $\mathbf{A}\ne\emptyset$ (see (Chentsov , 2008, part 2)). So, $\mathbf{A}\in\mathcal{P}'(\mathbf{P})$. From (1), it is evident that a route not defines the process development. Therefore, we introduce tracks or trajectories.

We suppose that under $j\in\overline{1,N}$

$$(\widehat{\mathbf{M}}_j := \{\mathrm{pr}_1(z) : z\in\mathbf{M}_j\})\&(\check{\mathbf{M}}_j := \{\mathrm{pr}_2(z) : z\in\mathbf{M}_j\}).$$

Moreover, suppose that

$$(\widetilde{\mathbf{X}} := X^0\bigcup(\bigcup_{j=1}^N\widehat{\mathbf{M}}_j))\&(\mathbf{X} := X^0\bigcup(\bigcup_{j=1}^N\check{\mathbf{M}}_j)).$$

By $\mathbf{Z}$ we denote the set of all processions $(z_i)_{i\in\overline{0,N}}$ : $\overline{0,N}\longrightarrow\widetilde{\mathbf{X}}\times\mathbf{X}$.

Under $x\in X^0$ and $\alpha\in\mathbf{P}$, we obtain that

$$\mathcal{Z}_\alpha[x] := \{(z_i)_{i\in\overline{0,N}}\in\mathbf{Z}|(z_0=(x,x))\&(z_t\in\mathbf{M}_{\alpha(t)}$$
$$\forall t\in\overline{1,N})\}\in\mathrm{Fin}(\mathbf{Z})$$

is the set of all tracks coordinated with the route $\alpha$. Of course, under $x\in X^0$

$$\mathbf{D}[x] := \{(\alpha,\mathbf{z})\in\mathbf{A}\times\mathbf{Z}|\mathbf{z}\in\mathcal{Z}_\alpha[x]\}\in\mathrm{Fin}(\mathbf{A}\times\mathbf{Z}),$$

is the set of all admissible solutions defined as OP route-track.

**Cost functions.** We consider non additive variant of the cost aggregation. In this connection, we suppose that $\widehat{N} := \mathcal{P}'(\overline{1,N})$ and fix $\mathbf{c}\in\mathcal{R}_+[\mathbf{X}\times\widetilde{\mathbf{X}}\times\widehat{N}]$, $c_1\in\mathcal{R}_+[\widetilde{\mathbf{X}}\times$

$\mathbf{X} \times \widehat{N}]$ $c_N \in \mathcal{R}_+[\widetilde{\mathbf{X}} \times \mathbf{X} \times \widehat{N}$, where $\widehat{N} := \mathcal{P}'(\overline{1,N})$. By $\mathbf{c}, c_1, ..., c_N$ cost functions are denoted.

Moreover, we fix a parameter $\mathbf{a} \in \mathbf{R}_+, \mathbf{a} \neq 0$. Now, we introduce the next nonadditive criterion. Namely, under $\alpha \in \mathbf{P}$ and $\mathbf{z} \in \mathbf{Z}$, let

$$\mathcal{B}_\alpha[\mathbf{z}] := \max_{t \in \overline{0,N-1}} \mathbf{a}^t[\mathbf{c}(\mathrm{pr}_2(\mathbf{z}(t)), \mathrm{pr}_1(\mathbf{z}(t+1)), \alpha^1(\overline{t+1,N}))$$
$$+c_{\alpha(t+1)}(\mathbf{z}(t+1), \alpha^1(\overline{t+1,N}))] \in \mathbf{R}_+; \quad (3)$$

For the following in (3), the case $\alpha \in \mathbf{A}$ and $\mathbf{z} \in \mathcal{Z}_\alpha[x]$, where $x \in X^0$, is essential. So, under $x \in X^0$, we obtain the problem

$$\mathcal{B}_\alpha[\mathbf{z}] \longrightarrow \min, (\alpha, \mathbf{z}) \in \mathbf{D}[x]; \quad (4)$$

for this problem (4), the value

$$V[x] := \min_{(\alpha,\mathbf{z}) \in \mathbf{D}[x]} \mathcal{B}_\alpha[\mathbf{z}] \in \mathbf{R}_+ \quad (5)$$

is defined and, moreover,

$$(\mathrm{SOL})[x] := \{(\alpha, \mathbf{z}) \in \mathbf{D}[x] | \mathcal{B}_\alpha[\mathbf{z}] = V[x]\} \in \mathcal{P}'(\mathbf{D}[x]).$$

Finally, it is important (see (5)) the following problem of the initial state optimization

$$V[x] \longrightarrow \inf, \quad x \in X^0. \quad (6)$$

For problem (6), the next extremum is defined:

$$\mathbf{V} := \inf_{x \in X^0} V[x].$$

## 4. DYNAMIC PROGRAMMING METHOD

This section is devoted to the dynamic programming method. The variant of DP proposed here is similar to Chentsov (2016). The considered variant of DP allows to reduce the performed calculations a little. We introduce essential lists of tasks and layers of the position space. Let

$$\mathcal{G} := \{K \in \widehat{N} | \forall z \in \mathbf{K}\ (\mathrm{pr}_1(z) \in K) \Rightarrow (\mathrm{pr}_2(z) \in K)\}. \quad (7)$$

We defined the set of all essential lists of tasks. Moreover,

$$\mathcal{G}_s := \{K \in \mathcal{G} | s = |K|\} \quad \forall s \in \overline{1,N}.$$

Notice, that $\mathcal{G}_N = \{\overline{1,N}\}$. Let $\mathbf{K}_1 := \{\mathrm{pr}_1(z) : z \in \mathbf{K}\}$; then $\mathcal{G}_1 = \{\{t\} : t \in \overline{1,N} \backslash \mathbf{K}_1\}$. Finally Chentsov (2013)

$$\mathcal{G}_{s-1} = \{K \backslash \{t\} : K \in \mathcal{G}_s,\ t \in \mathbf{I}(K)\} \quad \forall s \in \overline{2,N}.$$

As a result we have recurrent procedure

$$\mathcal{G}_N \longrightarrow ... \longrightarrow \mathcal{G}_1.$$

With the help of this recurrent procedure, we construct sets $D_0, D_1, ..., D_N$. We will call these sets layers of the position space. Suppose that

$$\widetilde{\mathcal{M}} := \bigcup_{j \in \overline{1,N} \backslash \mathbf{K}_1} \check{\mathbf{M}}_j$$

and $D_0 := \widetilde{\mathcal{M}} \times \{\emptyset\} = \{(x, \emptyset) : x \in \widetilde{\mathcal{M}}\}$. Moreover,

$$D_N := X^0 \times \{\overline{1,N}\} = \{(x, \overline{1,N}) : x \in X^0\}.$$

Then, $D_0$ and $D_N$ are extreme layers of the position space. Consider the construction procedure of immediate layers. Namely, at the beginning, for $s \in \overline{1,N-1}$ and $K \in \mathcal{G}_s$, suppose sequentially that

$$J_s(K) := \{j \in \overline{1,N} \backslash K | \{j\} \cup K \in \mathcal{G}_{s+1}\},$$

$$\mathcal{M}_s[K] := \bigcup_{j \in J_s(K)} \check{\mathbf{M}}_j,$$

$$\widetilde{\mathbf{D}}_s[K] := \mathcal{M}_s[K] \times \{K\} = \{(x, K) : x \in \mathcal{M}_s[K]\}.$$

We define

$$D_s := \bigcup_{K \in \mathcal{G}_s} \widetilde{\mathbf{D}}_s[K]\ \forall s \in \overline{1,N-1}.$$

In this way, we have the position sets of positions $D_0 \neq \emptyset, D_1 \neq \emptyset, ..., D_N \neq \emptyset$. Additionally, by (Chentsov, 2013, (6.11)) we have the property:

$$(\mathrm{pr}_2(z), K \backslash \{j\}) \in D_{s-1} \quad \forall s \in \overline{1,N}$$
$$\forall (x, K) \in D_s\ \forall j \in \mathbf{I}(K)\ \forall z \in \mathbf{M_j}. \quad (8)$$

As a result we construct sequentially the functions $v_0 \in \mathcal{R}_+[D_0]$, $v_1 \in \mathcal{R}_+[D_1], ..., v_N \in \mathcal{R}_+[D_N]$ namely, we nave a recurrent procedure

$$v_0 \longrightarrow v_1 \longrightarrow ... \longrightarrow v_N. \quad (9)$$

So, we define $v_0 \in \mathcal{R}_+[D_0]$ by the rule $v_0(x, \emptyset) := 0\, \forall x \in \widetilde{\mathcal{M}}$. If $s \in \overline{1,N}$ and $v_{s-1} \in \mathcal{R}_+[D_{s-1}]$ was already constructed, then (see (8)) we define $s_s \in \mathcal{R}_+[D_s]$ by the rule

$$v_s(x, K) = \min_{j \in \mathbf{I}(K)} \min_{z \in \mathbf{M}_j} \sup(\{\mathbf{c}(x, \mathrm{pr}_1(z), K)$$
$$+c_j(z, K); \mathbf{a}v_{s-1}(\mathrm{pr}_2(z), K \backslash \{j\})\}) \quad \forall (x, K) \in D_s. \quad (10)$$

Using (10) we will calculate all functions in (9).

Eventually, we obtain procession $(v_0, v_1, ..., v_N)$ including construction $v_N$. Note that $v_0$ is determined by the identity $v_0(x) \equiv 0$ and

$$V[x^0] = v_N(x^0, \overline{1,N}) \quad \forall x^0 \in X^0. \quad (11)$$

The property (11) is established similarly to (Chentsov , 2016, (4.15)) (we note that, in our case, the proposition similar to (Chentsov , 2016, Theorem 1) is valid; but, in this proposition, parameter $\mathbf{a}$ is used). Of course, by (10) and (11) we obtain that

$$V[x^0] = \min_{j \in \mathbf{I}(\overline{1,N})} \min_{z \in \mathbf{M}_j} \sup(\{\mathbf{c}(x^0, \mathrm{pr}_1(z), \overline{1,N})$$
$$+c_j(z, \overline{1,N}); \mathbf{a}v_{N-1}(\mathrm{pr}_2(z), \overline{1,N} \backslash \{j\})\}) \quad \forall x^0 \in X^0. \quad (12)$$

From (12) and (6) we have

$$v_N(x^0, \overline{1,N}) \longrightarrow \inf,\ x^0 \in X^0.$$

If $x^0 \in X^0$, then the construction of $(\alpha^0, z^0) \in (\mathrm{SOL})[x^0]$ is realized similarly to (Chentsov , 2016, (4.20)—(4.27)) with application of parameter $\mathbf{a}$; compare (10) and (Chentsov , 2016, Proposition 1).

## 5. OPTION METRIC SPACE $X^0$

Here we will assume that the set $X^0$ is a metric space with a norm $\rho$. For $x \in X^0$ and $\varepsilon \in \mathbf{R}_+$, $\epsilon \neq 0$,

$$B^0_\rho(x, \varepsilon) := \{y \in X^0 | \rho(x, y) < \varepsilon\} \in \mathcal{P}'(X^0)$$

is the corresponding open ball with the center $x$ and radius $\varepsilon$. Suppose that $(X^0, \rho)$ is completely bounded space, namely

$$\forall \varepsilon \in \mathbf{R}_+ \backslash \{0\}\ \exists K \in \mathrm{Fin}(X^0) : X^0 = \bigcup_{x \in K} B^0_\rho(x, \varepsilon). \quad (13)$$

Note that if $X^0$ is a bounded set in a finite-dimensional arithmetic space and $\rho$ is Euclidean metric on $X^0$, then (13) is fulfilled.

By $\widetilde{\mathbf{M}}$ we denote the union of all sets $\widehat{\mathbf{M}}_j,\ j \in \overline{1,N}$.

**Assumption 4.1.** $\forall \varepsilon \in \mathbf{R}_+,\ \epsilon \neq 0\, \exists \delta \in \mathbf{R}_+\, \delta \neq 0\ \forall x_1 \in X^0\ \forall x_2 \in X^0$

$$(\rho(x_1,x_2) < \delta) \Rightarrow (|\mathbf{c}(x_1,y,\overline{1,N}) - \mathbf{c}(x_2,y,\overline{1,N})| < \varepsilon \quad \forall y \in \widetilde{\mathbf{M}}).$$

We suppose that Assumption 4.1 is fulfilled.

**Proposition 4.1.** Let $\varepsilon^0 \in \mathbf{R}_+$ and $\epsilon^0 \neq 0$. Moreover, let $\delta^0 \in \mathbf{R}_+ \delta \neq 0$ such that

$$(\rho(x_1,x_2) < \delta^0) \Rightarrow (|\mathbf{c}(x_1,y,\overline{1,N}) - \mathbf{c}(x_2,y,\overline{1,N})| < \varepsilon_0 \quad \forall y \in \widetilde{\mathbf{M}}).$$

And also let $\mathcal{K} \in \mathrm{Fin}(X^0)$ is the set for which

$$X^0 = \bigcup_{x \in \mathcal{K}} B^0_\rho(x, \delta_0).$$

Then, we obtain two-sided estimation:

$$\mathbf{V} \le \min_{x \in \mathcal{K}} v_N(x, \overline{1,N}) \le \mathbf{V} + \varepsilon_0. \tag{14}$$

The correspondence proof realized by immediate combination of (12), (13), and Assumption 4.1.

So, in our case, for determination of the global extremum with every precision, discretizations of $X^0$ can be used. In addition, the universal variant of DP procedure similar to Chentsov (2016) can by applied (see (9), (10)).

Now, by Proposition 4.1 we can be restricted to consideration of variants for problems similar to (6) and using instead of $X^0$ finite subsets of $X^0$.

Do, we consider one such problem fixing $\mathcal{K} \in \mathrm{Fin}(X^0)$ (of course, we are oriented on realization of relations similar to (14)). We suppose that all functions (9) were constructed. We find $x_0 \in \mathcal{K}$ such that

$$v_N(x_0, \overline{1,N}) = \min_{x \in \mathcal{K}} v_N(x, \overline{1,N}) \tag{15}$$

(of course, $x_0 \in X^0$ and $V[x_0] = v_N(x_0, \overline{1,N})$). Now, we consider the Bellman procedure to calculation of solution of the set $(\mathrm{SOL})[x_0]$ (by analogy with the results from (Chentsov , 2016, §4)).

So, we suppose $\mathbf{z}^{(0)} := (x_0, x_0) \in \widetilde{\mathbf{X}} \times \mathbf{X}$ and choose (see (12)) $\mathbf{j}_1 \in \mathbf{I}(\overline{1,N})$ and $\mathbf{z}^{(1)} \in \mathbf{M}_{\mathbf{j}_1}$ for which

$$V[x_0] = \sup(\{\mathbf{c}(x^0, \mathrm{pr}_1(\mathbf{z}^{(1)}), \overline{1,N}) + c_{\mathbf{j}_1}(\mathbf{z}^{(1)}, \overline{1,N}); \mathbf{a} v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N} \setminus \{\mathbf{j}_1\})\}). \tag{16}$$

Then, according to (8) the next inclusion is performed where

$$(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N} \setminus \{\mathbf{j}_1\}) \in D_{N-1} \tag{17}$$

and $N - 1 \ge 1$. From (10) and (17) we have

$$v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N} \setminus \{\mathbf{j}_1\}) = \min_{j \in \mathbf{I}(\overline{1,N} \setminus \{\mathbf{j}_1\})} \min_{z \in \mathbf{M}_j} \sup(\{\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(z), \overline{1,N} \setminus \{\mathbf{j}_1\}) + c_j(z, \overline{1,N} \setminus \{\mathbf{j}_1\}); \mathbf{a} v_{N-2}(\mathrm{pr}_2(z), \overline{1,N} \setminus \{\mathbf{j}_1; j\})\}).$$

Using this equality, we choose $\mathbf{j}_2 \in \mathbf{I}(\overline{1,N} \setminus \{\mathbf{j}_1\})$ and $\mathbf{z}^{(2)} \in \mathbf{M}_{\mathbf{j}_2}$ for which

$$v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N} \setminus \{\mathbf{j}_1\}) = \sup(\{\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\mathbf{j}_1\}) + c_{\mathbf{j}_2}(\mathbf{z}^{(2)}, \overline{1,N} \setminus \{\mathbf{j}_1\}); \mathbf{a} v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\mathbf{j}_1; \mathbf{j}_2\})\}), \tag{18}$$

where by (8) $(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\mathbf{j}_1; \mathbf{j}_2\}) \in D_{N-2}$. From (16) and (18) we obtain the following equality:

$$V[x_0] = v_N(x_0, \overline{1,N}) = \sup(\{\max_{t \in \overline{0,1}} \mathbf{a}^t [\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(t)}), \mathrm{pr}_1(\mathbf{z}^{(t+1)}), \overline{1,N} \setminus \{\mathbf{j}_k : k \in \overline{1,t}\}) + c_{\mathbf{j}_{t+1}}(\mathbf{z}^{(t+1)}, \overline{1,N} \setminus \{\mathbf{j}_k : k \in \overline{1,t}\})]; \mathbf{a}^2 v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\mathbf{j}_k : k \in \overline{1,2}\})\}), \tag{19}$$

where the obvious equality $\overline{1,0} = \emptyset$ is used. Under $N = 2$, by (19), we obtain the optimal solution $((\mathbf{j})_{k \in \overline{1,2}}, (\mathbf{z}^{(k)})_{k \in \overline{0,2}})$. For $N > 2$, it is required to continue the choice procedure similar to (16) and (18) intil exhaustion of the index set $\overline{1,N}$. Then, we obtain $\mathbf{j}_1 \in \overline{1,N}, ..., \mathbf{j}_N \in \overline{1,N}$, $\mathbf{z}^{(0)} \in \widetilde{\mathbf{X}} \times \mathbf{X}, \mathbf{z}^{(1)} \in \widetilde{\mathbf{X}} \times \mathbf{X}, ..., \mathbf{z}^{(N)} \in \widetilde{\mathbf{X}} \times \mathbf{X}$ for which

$$\eta_0 := (\mathbf{j}_k)_{k \in \overline{1,N}} \in \mathbf{A},\ \mathbf{z}_0 = (\mathbf{z}^{(k)})_{k \in \overline{0,N}} \in \mathcal{Z}_{\eta_0}[x_0],$$

and $\mathcal{B}_{\eta_0}[\mathbf{z}_0] = v_N(x_0, \overline{1,N}) = V[x_0]$. So, $(\eta_0, \mathbf{z}_0) \in (\mathrm{SOL})[x_0]$.

Proposition 4.1 determines the theoretical possibility of discretizing the set $X^0$ from the considerations of a given accuracy for the realization of $\mathbf{V}$. In the next section, the results of a computational experiment are presented in the case when the set $X^0$ itself is finite and its discretization is not required. Thus, the constructions of the next section can be considered independently of Proposition 4.1.

## 6. EXAMPLES

Below we present the results of mathematical modeling of the developed optimal algorithm. In the considered examples, the problem of bypassing 31 megalopolises on a plane is considered. As megalopolises, we consider a set of twelve points uniformly located on circles. Sources of radiation are within the circumference. We consider 34 address pairs as precedence conditions. Four possible starting points (-70, -95), (0,0), (80, -90), (90.35) are given, from which the route and the traversal path of the sets can be read. The cost function $\mathbf{c}$ of external displacements was determined using Euclidean distance. We also assume that the speed of movement outside megalopolises is 4 times greater than that within megalopolises.

Functions $c_1, ..., c_N$ that evaluated the costs when performing work on target sets (megalopolises) were determined each time by summing two Euclidean distances: from the point of arrival to a fixed point and connected to a megalopolis, and from this point to the point of departure. We assume that when a $M_j$, where $j \in \overline{1,N}$ megalopolis is visited, the performer selects the arrival point (among the points $M_j$) from which it is directed to the point $m_j$ that belongs to the convex hull $M_j$, performs the required operation there, and then moves to the departure point.

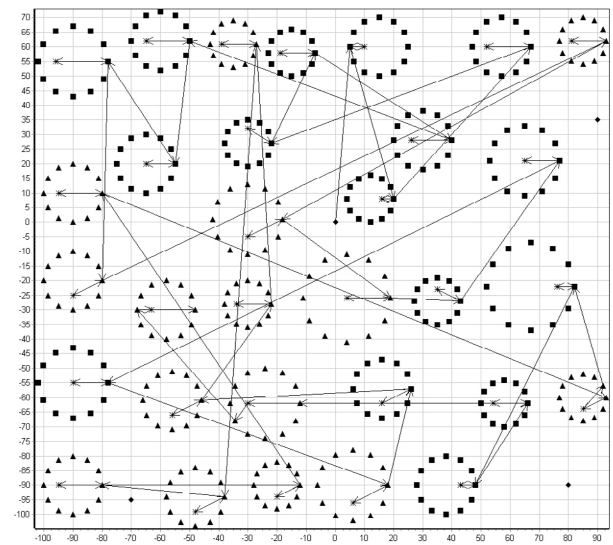

Fig. 1. Variant of displacements at $\mathbf{a} = 1$

As $\mathbf{M}_j$, we use the Cartesian product $M_j \times M_j$ each time. The term "total costs" below is the value of $\mathbf{V}$. The algorithm ensures its finding and constructing the optimal permissible solution in the form of a set consisting of an initial state, a route and a trajectory.

We are considering a generalized version of the bottleneck routing problem, for which a global extremum and an optimal solution are determined. The calculations were carried out for three values of the parameter $\mathbf{a}$ : $\mathbf{a} = 1$ (the usual bottleneck routing problem), $\mathbf{a} = 1,1$ (a problem in which the final fragments of the solution, defined in the form of a route-trajectory pair, are more significant), $\mathbf{a} = 0,9$ (the case of the main problem, when the initial fragments of the solution are more significant).

The software product is written in the programming language C ++, running in the 64-bit operating system of the Windows family, starting with Windows 7. The computational part of the program is implemented in a separate stream from the user interface. For the case of solving a problem on a plane, it is possible to graphically represent the route and the route and increase the individual sections of the graph; the image can be saved to a bmp image file.

We conducted a computational experiment on a computer with a central processor Intel Core i7, 64 GB of RAM and the operating system Windows 7 Maximum SP1.

The results of the computational experiment are shown in the following three figures. The exit point from the last megalopolis is given as the final point.

**Variant a = 1.** "Total costs": 22,845. Starting point $x^0 = (0, 0)$. Final point — $(-68, -30)$. Time of calculation — 5 hours 11 min. 20 seconds.

**Variant a = 1,1.** "Total costs": 132,84. Starting point $x^0 = (-70, -95)$. Final point — $(-78, -55)$. Time of calculation — 5 hours 11 min. 29 seconds.

**Variant a = 0.9.** "Total costs": 10,3. Starting point $x^0 = (0, 0)$. Final point — $(25, 60)$. Time of calculation — 5 hours 12 min. 4 seconds.

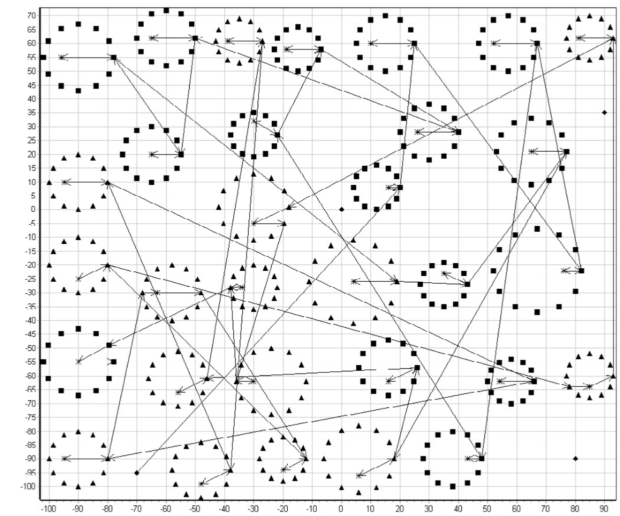

Fig. 2. Variant of displacements at $\mathbf{a} = 1.1$

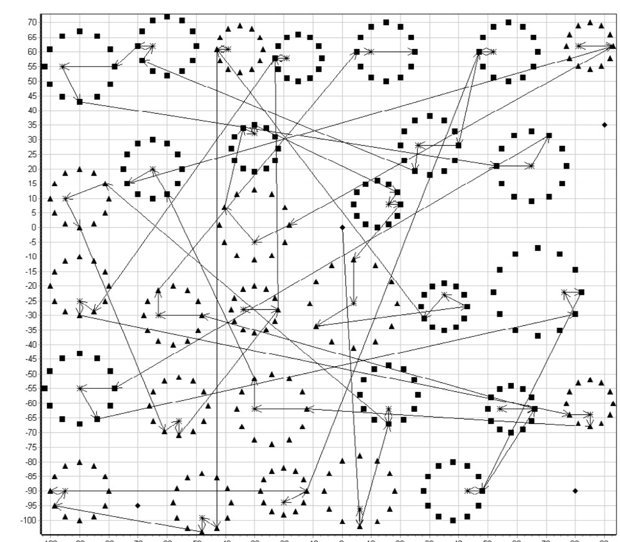

Fig. 3. Variant of displacements at $\mathbf{a} = 0.9$

## 7. CONCLUSION

The developed algorithm for determining the optimal sequence of performing radiation-hazardous activities in a contaminated area allows determining the sequence of bypassing radioactive objects to be eliminated, at which the maximum daily radiation dose over the entire period of work will take the smallest value. The proposed algorithm is based on a dynamic programming option. A feature of this variant of dynamic programming is that the criterion is non-additive, the objective function depends on the list of outstanding work. In addition, the algorithm allows you to optimally choose the starting point. The effectiveness of the proposed algorithm is illustrated by the example.

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